

**EXTERIOR DIFFERENTIAL SYSTEMS:
A GEOMETRIC APPROACH TO PDE
LECTURE NOTES FROM THE 1997 DAEWOO WORKSHOP**

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LECTURE 1: INTRODUCTION TO EXTERIOR DIFFERENTIAL SYSTEMS.

In this lecture we will see how to study a system of partial differential equations (henceforth denoted pde) from a geometric perspective.

Example 1, the minimal surface equation. Given a surface in Euclidean 3-space, $M^2 \subset \mathbb{E}^3$, described locally as a graph $z = z(x, y)$, at a point $p = (x, y, z(x, y))$, let

$$H(p) = \frac{1}{2} \frac{(1 + z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1 + z_x^2)z_{yy}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}.$$

$H(p)$ is called the *mean curvature function* of the surface M . $H(p)$ has geometric meaning, that is, it is a well defined function on M , independent of coordinates chosen and invariant under the displacement of M by rigid motions.

Consider the pde for surfaces in \mathbb{E}^3 with the property that $H(p) \equiv 0$. Geometrically such surfaces are critical for the variation of area and are called *minimal surfaces*. If M is a minimal surface, then for all $x \in M$, there exists a open neighborhood $\tilde{U}_x \subset M$ such that for all open subsets $U \subset \tilde{U}_x$ and patches of surface V with $\partial V = \partial U$, then $\text{area}(V) \geq \text{area}(U)$ (see e.g. [S, III]).

We will be studying minimal surfaces and higher dimensional generalizations in the next lecture. One amazing aspect of the pde for minimal surfaces in \mathbb{E}^3 is that the solutions are exactly given by solutions to the Cauchy-Riemann equations. More precisely, there is the following theorem:

Theorem (Weierstrass [S, IV, p 395]). *Minimal surfaces $M^2 \subset \mathbb{E}^3$ can be described by holomorphic functions. More precisely, every minimal surface in \mathbb{E}^3 is locally of the form:*

$$\begin{aligned} x(w, \bar{w}) &= \text{Re} \frac{1}{2} \int f(w)(1 - g(w)^2)dw \\ y(w, \bar{w}) &= \text{Re} \frac{i}{2} \int f(w)(1 + g(w)^2)dw \\ z(w, \bar{w}) &= \text{Re} \int f(w)g(w)dw \end{aligned}$$

where $g(w)$ is a meromorphic function and $f(w)$ is a holomorphic function vanishing precisely at the poles of g and the integrals are path integrals beginning at some fixed point w_0 to the point w .

There is actually ambiguity in the formulae and, as we will see, the minimal surfaces essentially depend on one holomorphic function.

How could one discover such a beautiful theorem? We will see that by rephrasing the minimal surface equation as an exterior differential system, one might guess that such a formula exists, and in fact be able to see the geometry of such. (Of course Weierstrass had to work a lot harder since he didn't have the machinery available to him.) The exterior differential systems machinery has been used to recognize other systems of pde appearing in geometry as the Cauchy-Riemann system and other familiar systems.

The minimal surfaces $M^2 \subset \mathbb{E}^{2+s}$, are also all described in terms of holomorphic functions but this fails to be true when $\dim M > 2$.

On the other hand, one might hope that special classes of minimal submanifolds might still be described by holomorphic data. (This is because the minimal submanifold equation “looks like” Laplace’s equation (they have the same symbol) so just as in any even dimension one can find special solutions of Laplace’s equation by solving the Cauchy-Riemann equations (or more generally solutions of a Dirac equation) one could hope to do the same for minimal submanifolds. When we have understood goal 1 below, we will see how to arrive at the description of minimal surfaces in terms of holomorphic data and how to look for special classes of minimal submanifolds of arbitrary dimension.

We will have the following general goals for these lectures:

1. *To explain how to find an appropriate geometric setting for studying a given system of pde.*

To do this we will first work on a larger space to eliminate choices of coordinates that have no geometric meaning, and then we will “quotient out” to a smaller space where variables not relevant to the system are eliminated.

On the geometrically determined space one can then hope to recognize familiar systems in disguise, and to find special classes of solutions (e.g. solutions with “symmetry”).

2. *To explain the Cartan algorithm to determine the moduli space of local solutions to any given exterior differential system and an appropriate initial value problem for the system.*

The general Cartan algorithm is somewhat complicated to state, so for the moment, I’ll just mention a theorem we will study in lecture 3 that is proved using the algorithm:

BCJS Theorem (see [BCG³, p 302],[S, V, p 216]). *Let g be any analytic Riemannian metric in a neighborhood V of $0 \in \mathbb{R}^n$. There exist local analytic isometric embeddings on some smaller neighborhood $U \subseteq V$, $i : (U, g) \rightarrow \mathbb{E}^{\binom{n+1}{2}}$. The embeddings depend on n functions of $n - 1$ variables.*

Note the form of the conclusion of the BCJS theorem. This is the type of answer the Cartan-Kähler theorem provides, not only an assertion of existence, but a rough description of the size of the moduli space of solutions.

Recently I have utilized the Cartan machinery to study questions in algebraic geometry. It has been quite useful for sorting out what of the geometry of projective varieties arises from global considerations and what is due to the local projective geometry. (See [L, 3-8]).

Linear Algebra Aside. Let V be a vector space (\mathbb{R}^n or \mathbb{C}^n for our purposes), and let V^* denote the dual vector space. Let $\theta^1, \dots, \theta^s \in V^*$ be linearly independent. One has the correspondence

$$\begin{array}{ccc} s - \text{planes in } V^* & \leftrightarrow & (n - s) - \text{planes in } V \\ \text{span}\{\theta^1, \dots, \theta^s\} & & \{v \in V | \theta^a(v) = 0, 1 \leq a \leq s\} \end{array}$$

Example 2. Consider the system of pde

$$(2.1) \quad u_x = A(x, y, u) \quad u_y = B(x, y, u)$$

for one function of two variables (where A, B are given functions). Note that it is over-determined (two equations for one function) so one expects that there won't be any solutions unless A and B satisfy a compatibility condition. To phrase this system geometrically, let \mathbb{R}^3 have coordinates (x, y, u) and consider the forms

$$\theta := du - Adx - Bdy \quad \Omega := dx \wedge dy$$

Observe that we have the correspondence

$$\begin{aligned} \text{Solutions to (2.1)} &\leftrightarrow \text{surfaces } i : S \rightarrow \mathbb{R}^3, \text{ such that} \\ i^*(\theta) &\equiv 0 \text{ and } i^*(\Omega) \text{ is nonvanishing.} \end{aligned}$$

When using the exterior differential systems machinery, the problem of finding solutions to a system of pde is always rephrased as a problem in submanifold geometry as in this example.

Geometrically, at each $p \in \mathbb{R}^3$ there exists a unique two-plane $U_p \subset T_p \mathbb{R}^3$ annihilated by θ so if there exists a solution passing through p , its tangent plane must be U_p . So the question is: Do these two-planes "fit together"? This is a second order condition. To determine the condition in coordinates, we utilize that mixed partials commute:

$$\begin{aligned} (u_x)_y &= \frac{\partial}{\partial y} A(x, y, u) = A_y(x, y, u) + \frac{\partial u}{\partial y} A_u(x, y, u) = A_y + BA_u \\ (u_y)_x &= B_x + AB_u \end{aligned}$$

Setting $(u_x)_y = (u_y)_x$ uncovers the equation

$$(2.2) \quad A_y + BA_u = B_x + AB_u$$

More geometrically, $(u_x)_y = (u_y)_x$ if and only if

$$(2.3) \quad d\theta \equiv 0 \bmod \{\theta\}$$

Notation: If v_1, \dots, v_s are vectors, $\{v_1, \dots, v_s\}$ denotes their linear span. If $\theta^1, \dots, \theta^s \in \Omega^*(M)$ are forms of any degree, let $\mathcal{I} = \{\theta^1, \dots, \theta^s\}_{alg}$ denote the algebraic ideal generated by $\theta^1, \dots, \theta^s$. Given $\beta \in \Omega^k(M)$, we say $\beta \equiv 0 \bmod \{\theta^1, \dots, \theta^s\}_{alg}$ if $\beta = \alpha_1 \wedge \theta^1 + \dots + \alpha_s \wedge \theta^s$ for some $\alpha_1, \dots, \alpha_s$.

Clearly (2.3) is a necessary condition, but in fact it is sufficient.

Frobenius Theorem (see [BCG³, p 27]). Let $\theta^1, \dots, \theta^s$ be a pointwise linearly independent one-forms on a manifold Σ^{n+s} of dimension $n+s$. If for all $x \in \Sigma$ and for all $1 \leq a \leq s$, we have

$$(2.4) \quad d\theta^a \equiv 0 \bmod \{\theta^1, \dots, \theta^s\}$$

then through each $x \in \Sigma$ there exists a unique submanifold $i : M \rightarrow \Sigma$ such that for all $y \in M$, $T_y M = \{\theta_y^1, \dots, \theta_y^s\}^\perp$. In this case we say the distribution $\{\theta^1, \dots, \theta^s\}^\perp$ is integrable. (${}^\perp$ denotes the annihilator in the dual vector space.)

For example 2, there are the following cases:

Case 1. The system is integrable.

Case 2. Otherwise, to find a solution, restrict to $\Sigma' \subset \mathbb{R}^3$, the surface where (2.2) holds. In the neighborhood of a $p \in \Sigma'$ there are two possibilities.

Case 2a. Since $\dim \Sigma' = 2$, we already have a surface so if u appears in the equation defining Σ' , equivalently, if $i^*(\Omega)$ is nonvanishing, then solving for u gives a candidate for a solution and we need to check if this function satisfies the system, equivalently, if $i^*(\theta) = 0$.

Case 2b. If u does not appear, then on Σ' there is a relation between our independent variables x and y so $i^*(\Omega) = 0$ and there are no solutions.

Definitions: An *exterior differential system with independence condition* on a manifold Σ consists of an ideal $\mathcal{I} \subset \Omega^*(\Sigma)$ closed under exterior differentiation and a differential n -form $\Omega \in \Omega^n(\Sigma)$. ($\Omega^*(\Sigma) = \bigoplus \Omega^k(\Sigma)$ denotes the space of all differential forms on Σ .)

An *integral manifold* (solution) of the system (\mathcal{I}, Ω) is an immersed n -fold $f : M^n \rightarrow \Sigma$ such that $f^*(\alpha) = 0 \forall \alpha \in \mathcal{I}$ and $f^*(\Omega)$ is nonvanishing.

For example, in example 2, $\mathcal{I} = \{\theta\}_{\text{diff}} = \{\theta, d\theta\}_{\text{alg}}$. One could rephrase (2.4) as saying if $\{\theta^1, \dots, \theta^s\}_{\text{alg}} = \{\theta^1, \dots, \theta^s\}_{\text{diff}}$, then the system is integrable.

It is useful to study integral manifolds infinitesimally. Let V be a vector space, and let $G(n, V)$ denote the Grassmannian of n -planes through the origin in V . We define an *integral element* of (\mathcal{I}, Ω) at $x \in \Sigma$ to be an $E \in G(n, T_x \Sigma)$ such that $\alpha|_E = 0 \forall \alpha \in \mathcal{I}$ and $\Omega|_E \neq 0$. Integral elements are the candidates for tangent spaces to integral manifolds.

In these lectures we will actually only study a special class of exterior differential systems with independence condition. A *Pfaffian* eds is one generated by differentially one-forms, say θ^a , $1 \leq a \leq s$. I.e. $\mathcal{I} = \{\theta^a\}_{\text{diff}} = \{\theta^a, d\theta^a\}_{\text{alg}}$. Let $I_x := \{\theta^a|_x\} \subset T_x^*\Sigma$ be the distribution they generate. Write the independence condition as $\Omega = \omega^1 \wedge \dots \wedge \omega^n$, and let $J_x := \{\theta^a|_x, \omega^i|_x\}$. The Pfaffian system is *linear* if

$$d\theta^a \equiv 0 \pmod{J}.$$

We will restrict our attention to linear Pfaffian systems. The name “linear” is used because if a Pfaffian system is linear, then its integral elements at a point are determined by linear equations, see e.g. example 3 below. One often denotes a linear Pfaffian system by (I, J) instead of (\mathcal{I}, Ω) .

In principle we can study any exterior differential system with independence condition just by studying linear Pfaffian systems as follows:

The canonical linear Pfaffian system on $G(n, T\Sigma)$. Let Σ be any manifold. Consider $\pi : G(n, T\Sigma) \rightarrow \Sigma$, the Grassmann bundle of all n -planes in all tangent spaces to Σ . We will denote points of $G(n, T\Sigma)$ by (p, E) where $p \in \Sigma$ and $E \subset T_p \Sigma$ is an n -dimensional subspace. $G(n, T\Sigma)$ carries a canonical linear Pfaffian system (I, J) on it. Namely it is the system whose solutions are exactly the lifts to $G(n, T\Sigma)$ of mappings $f : X^n \rightarrow M$, where the lift of f is $\tilde{f}(x) = (f(x), T_{f(x)} f(X))$. The system is defined by

$$I_{(p, E)} := \pi^*(E^\perp) \quad J_{(p, E)} := \pi^*(T_p^*\Sigma)$$

The prolongation of an exterior differential system. Now say Σ is equipped with an exterior differential system with independence condition (\mathcal{I}, Ω) . Let

$$\tilde{\Sigma} := \{(x, E) \in G(n, T\Sigma) \mid E \in G(n, T_x \Sigma) \text{ is an integral element of } (\mathcal{I}, \Omega) \text{ at } x\}.$$

The pullback of the canonical system (I, J) on $G(n, T\Sigma)$ to $\tilde{\Sigma}$ is a linear Pfaffian system called the *prolongation* of (\mathcal{I}, Ω) .

Any first order system of pde for maps $u : \mathbb{R}^n \rightarrow \mathbb{R}^s$ may be described as a linear Pfaffian system as follows:

The canonical contact system on the space of one-jets. Let $J^1 = J^1(\mathbb{R}^n, \mathbb{R}^s) := \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$ with coordinates (x^i, u^a, p_i^a) . J^1 is called the *space of one-jets of mappings from \mathbb{R}^n to \mathbb{R}^s* . On J^1 let

$$(j1.1) \quad \begin{aligned} \theta^a &:= du^a - p_i^a dx^i, \quad 1 \leq a \leq s \\ \Omega &:= dx^1 \wedge \dots \wedge dx^n. \end{aligned}$$

Integral manifolds of the system $(I := \{\theta^a\}, J := \{\theta^a, dx^i\})$ project to graphs of maps $u : \mathbb{R}^n \rightarrow \mathbb{R}^s$. We call the system (I, J) the *canonical contact system on J^1* . One can also work in the complex category with holomorphic functions and then we will call such a system a *complex contact system*.

Note that integral manifolds of (j1.1) are describable in terms of s arbitrary functions of n variables. Namely choose functions $f^a(x^1, \dots, x^n)$, $1 \leq a \leq s$ and set

$$\begin{aligned} u^a &= f^a(x^1, \dots, x^n) \\ p_i^a &= \frac{\partial f^a}{\partial x^i}(x^1, \dots, x^n). \end{aligned}$$

If one is ever so lucky to recognize a given system of pde as a contact system in disguise, one can simply write down local solutions in terms of arbitrary functions.

How to express a system of pde as a linear Pfaffian system. Now say we have a first order system of pde:

$$(j1.2) \quad F^r(x^i, u^a, \frac{\partial u^a}{\partial x^i}) = 0, \quad 1 \leq r \leq R$$

(R equations). Consider the codimension R submanifold $\Sigma \subset J^1$ given by $F^r(x^i, u^a, p_i^a) = 0$, $1 \leq r \leq R$. Solutions to (j1.2) correspond to integral manifolds of the restriction of the contact system on J^1 to Σ .

Note that lifting up to J^1 has the effect of turning the derivatives into independent variables. Restricting to Σ has the effect of forcing the pde to hold as an algebraic (or analytic) relation among the variables. The integral submanifolds of Σ are those where the variables representing the derivatives behave like derivatives and the independent variables stay independent.

Example 3, the Cauchy-Riemann equations. Write the Cauchy-Riemann equations as $u_{x^1}^1 = u_{x^2}^2, u_{x^2}^1 = -u_{x^1}^2$. Let $J^1(\mathbb{R}^2, \mathbb{R}^2)$ have coordinates (x^i, u^a, p_i^a) , $1 \leq i, j \leq 2, 1 \leq a, b \leq 2$, let $\Sigma^6 \subset J^1$ be defined by the equations $p_1^1 - p_2^2 = 0, p_2^1 + p_1^2 = 0$. The restriction of the canonical contact system (I, J) on J^1 to Σ is equivalent to the Cauchy-Riemann equations.

Following example 2, and having no idea what else to do, we differentiate. We have

$$\begin{aligned} d \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} &= \begin{pmatrix} -dp_1^1 & -dp_2^1 \\ -dp_1^2 & -dp_2^2 \end{pmatrix} \wedge \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix} \\ &= \begin{pmatrix} \pi_1 & \pi_2 \\ -\pi_2 & \pi_1 \end{pmatrix} \wedge \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}. \end{aligned}$$

where we use the notation $\pi_1 = -dp_1^1, \pi_2 = -dp_2^1$.

Note that $\theta^1, \theta^2, \pi_1, \pi_2, dx^1, dx^2$ form a basis of $T_x^*\Sigma$ for all $x \in \Sigma$ and the space of integral elements at x is given by the linear equations:

$$\begin{aligned} \theta^1 &= 0 \\ \theta^2 &= 0 \\ \pi_1 &= adx^1 + bdx^2 \\ \pi_2 &= bdx^1 - adx^2 \end{aligned}$$

where a, b are constants. Thus there is a two dimensional space of integral elements through each point, in contrast to the case of the Frobenius theorem where there was a unique integral element passing through each point.

Since Σ is a linear space of even dimension it can be given a complex structure, but what is important here is that it can be given a complex structure such that \mathcal{I} becomes a *complex contact structure*.

On $T^*\mathbb{C}\Sigma$, let

$$\begin{aligned}\theta &:= \theta^1 + i\theta^2 \\ \omega &:= dx^1 + idx^2 \\ \pi &:= \pi_1 + i\pi_2\end{aligned}$$

define an almost complex structure on Σ by specifying a basis of $T^{*10}\Sigma$. This almost complex structure is integrable. Moreover the system can be written as $\mathcal{I} = \{\theta, \bar{\theta}\}_{diff}$, $\Omega = \omega \wedge \bar{\omega}$.

(\mathcal{I}, Ω) is a *complex contact system* in the sense that there exist complex coordinates (w, p, z) on Σ such that $\mathcal{I} = \{\theta := dw - pdz, \bar{\theta} := d\bar{w} - \bar{p}d\bar{z}\}$ and integral manifolds are obtained by letting $f(z)$ be any holomorphic function and setting

$$w = f(z), \quad p = f'(z).$$

In the next lecture we will see how to recognize the eds for minimal surfaces as being equivalent to the Cauchy-Riemann equations. We will assume some familiarity with moving frames so those of you not yet familiar with moving frames are encouraged to do the preparatory exercises at the end of this lecture.

So far we have seen how to enlarge the space we work on, as stated in goal 1. Now I would like to describe how to “quotient out” to a smaller space.

Definition. Let \mathcal{I} be an exterior differential system on a manifold Σ . A vector field $\xi \in \Gamma(TM)$ is said to be a *Cauchy characteristic vector field* if $\xi \lrcorner \alpha = 0$ for all $\alpha \in \mathcal{I}$.

Proposition [BCG³, p 21]. *Let ξ_1, \dots, ξ_k be Cauchy characteristic vector fields, then the distribution they span is integrable.*

Thus if \mathcal{I} on Σ has Cauchy characteristic vector fields then we can actually work on a smaller space, “ Σ/leaves ”. We will see several examples of systems with Cauchy characteristic vector fields in the next two lectures.

In the few minutes remaining, I would like to mention a few results I obtained recently using the Cartan algorithm. Motivated by some problems in algebraic geometry, I addressed the following question:

How many derivatives does one need to take to see if a submanifold of affine or projective space is built out of linear spaces?

For example, given a surface, even in Euclidean 3-space, if it is negatively curved, then at each point there will be two lines osculating to order two (called the *asymptotic lines*). Thus to see if a surface in 3-space is ruled by lines one needs at least three derivatives. It is a classical result that in fact three derivatives are enough. Here is a generalization of this result:

Theorem [L7, 1]. *Let $X^n \subset \mathbb{A}^{n+a}$ or $X^n \subset \mathbb{P}^{n+a}$ be a patch of a smooth submanifold of an affine or projective space such that at every point there is a line osculating to order $n+1$. Then X is ruled by lines.*

There exist patches $X^n \subset \mathbb{A}^{n+1}$ or $X^n \subset \mathbb{P}^{n+1}$ having a line osculating to order n at every point that are not ruled. In fact over \mathbb{C} , every hypersurface has this property.

Given integers (n, a, k) one could ask what is the smallest number m_0 such that if $X^n \subset \mathbb{P}^{n+a}$ is a patch such that at each point there is a k plane osculating to order m_0 , then the k -planes are (locally) contained in X , yet there exist examples of patches $X^n \subset \mathbb{P}^{n+a}$ with k -planes osculating to orders $m_0 - 1$ that are not contained. Here is a partial result regarding this question:

Expectation Theorem, [L7, 6]. Let (n, a, k, m) be natural numbers satisfying $m \geq 3$ and

$$a[\binom{k+m-1}{m-1} - k-1] \geq k(n-k).$$

Let $X^n \subset \mathbb{A}^{n+a}$ or $X^n \subset \mathbb{P}^{n+a}$ be a patch of a smooth submanifold of an affine or projective space having the properties that at each $x \in X$ there exists a k -dimensional linear space L_x , disjoint from the fiber of the Gauss map, osculating to order m and such a certain rank condition holds through level $m-1$. Then these linear spaces are locally contained in X . If F denotes the fiber of the Gauss map at x and $\dim L_x \cap F = \lambda$, then the same conclusion holds as above with k, n respectively replaced by $k-\lambda, n-\lambda$. In fact, if the Gauss map is degenerate, one can replace L by the span of L and F .

The rank condition essentially requires that the differential invariants of X at x are reasonably generic among all possible tensors of differential invariants allowing a k -plane to osculate to order m , see [L7] for details.

I do not know if the m furnished by the expectation theorem is optimal. I showed it gave a lower bound when $k=1$ and was optimal when $k=1$ and $a=1$ and that it gave an upper bound when $k=n-2$. I am currently working on this question.

Problems for lecture 1.

- 1a. Calculate the dimension of the space of integral elements at a point of J^1 for the canonical contact system on J^1 .
- 1b. Calculate the dimension of the space of integral elements at a point of $\Sigma \subset J^1(\mathbb{R}^2, \mathbb{R}^1)$ where Σ corresponds to the hypersurface induced by the equation $u_x + u_y = 0$.
- 1c. What can one say about the dimension of the space of integral elements at a point of a hypersurface $\Sigma \subset J^1$ (where one restricts the canonical system to the hypersurface)?
2. Describe the construction corresponding to J^1 for second order systems of pde, then systems of k -th order. In each case, calculate the dimension of the space of integral elements of the canonical contact system.

Problems to prepare for lectures 2 and 3.

1. Let $G \subset Gl(n, \mathbb{R})$ be a matrix Lie group. Consider $g \in G$ as a map $g : G \rightarrow n \times n$ matrices. Let g^{-1} be the inverse matrix of g . Let $\Omega := g^{-1}dg$ be the corresponding matrix valued one-form. Ω is called the *Maurer-Cartan form* of G . Show that in fact it is \mathfrak{g} valued, where \mathfrak{g} is the Lie algebra of G . Show Ω satisfies the *Maurer-Cartan equation*, $d\Omega = -\Omega \wedge \Omega$. (Hint: calculate $0 = d(Id) = d(g^{-1}g)$.)
2. Prove the *Cartan lemma*: Let v_1, \dots, v_k be linearly independent elements of a vector space V and let w_1, \dots, w_k be elements of V such that $w_1 \wedge v_1 + \dots w_k \wedge v_k = 0$. Then there exist constants $h_{ij} = h_{ji}$, $1 \leq i, j \leq k$ such that $w_i = \sum_j h_{ij}v_j$.
3. Let (M^n, g) be a Riemannian manifold. Let $\mathcal{F} \rightarrow M$ denote the bundle of all orthonormal frames. Write $f \in \mathcal{F}$ as $f = (x, e_1, \dots, e_n)$, where $x \in M$ and e_1, \dots, e_n is an orthonormal basis of $T_x M$. Show that we may write $dx = \omega^i e_i$, $d\omega^i = \omega^j \omega^i_j$ where $\omega^i, \omega^i_j \in \Omega^1(\mathcal{F})$, with $\omega^i_j + \omega^j_i = 0$. The forms ω^i_j are called connection forms. In fact $\nabla_{e_i} X = dX(e_i) + \omega^j_i(X)e_j$, where ∇ is the Levi-Civita connection. Show that we have the equations

$$\begin{aligned} d\omega^i &= -\omega^i_j \wedge \omega^j \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + R^i_{jkl} \omega^l \end{aligned}$$

where the R^i_{jkl} are functions defined on \mathcal{F} such that $R^i_{jkl} e_i \otimes \omega^j \otimes \omega^k \otimes \omega^l$ descends to be a well defined element of $\Gamma(TM \otimes T^{*\otimes 3}M)$ which is the Riemann curvature tensor.

4. Let V be a vector space, let $G(n, V)$ denote the Grassmannian of n -planes through the origin in V . Show that for all $E \in G(n, V)$, the tangent space to E has the additional structure as a vector space of linear maps, more precisely, that $T_E G(n, V) \simeq E^* \otimes V/E$. Hint: Write $E = v_1 \wedge \dots \wedge v_n$, consider a curve $E(t) = v_1(t) \wedge \dots \wedge v_n(t)$ and differentiate at $t = 0$. If V comes equipped with an inner product, we may identify V/E with $E^\perp \subset V$.

LECTURE 2: APPLICATIONS TO THE STUDY OF MINIMAL SUBMANIFOLDS.

The frame bundle. It is not usually possible to choose coordinates adapted everywhere to the geometry of a given problem, but it is usually possible to choose adapted frames.

Let $V = \mathbb{R}^{n+s}$ have the standard inner product (we write V to emphasize the structure of a vector space to avoid confusion with Euclidean space \mathbb{E}^{n+s} .) Let $\mathcal{F}_{\mathbb{E}^{n+s}}$ be the bundle of all orthonormal framings of \mathbb{E}^{n+s} , that is each $f \in \mathcal{F}$ is $f = (x, u)$, where $x \in \mathbb{E}^{n+s}$ and $u : T_p \mathbb{E}^{n+s} \rightarrow V$ is an isometry.

Given a submanifold $M^n \subset \mathbb{E}^{n+s}$, we can define a subbundle of the restriction of \mathcal{F} to M , namely, writing $u = (e_1, \dots, e_n, e_{n+1}, \dots, e_{n+s})$, we may restrict further to frames such that $T_x M = \{e_1, \dots, e_n\}$. We denote this bundle by $\mathcal{F}^1 \rightarrow M$.

The main advantage of frame bundles is that they come equipped with a canonical framing, that is a canonical basis of their cotangent space at each point.

Use index ranges $1 \leq i, j, k \leq n$, $n + 1 \leq a, b \leq s$. On $\mathcal{F} \rightarrow \mathbb{E}^{n+s}$ we may write:

$$d(x, e_1, \dots, e_{n+s}) = (x, e_1, \dots, e_{n+s}) \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega_j^i & \omega_b^i \\ \omega^a & \omega_j^a & \omega_b^a \end{pmatrix}$$

where $\omega^\alpha, \omega_\beta^\alpha = -\omega_\alpha^\beta \in \Omega^1(\mathcal{F})$.

On $\mathcal{F}^1 \rightarrow M$, we have $\omega^a = 0$, which of course implies that $d\omega^a = 0$. (Here and in what follows we commit a standard abuse of notation, we really mean the pullback of ω^a to \mathcal{F}^1 is zero.) Since \mathcal{F} is a Lie group, letting Ω denote its Maurer-Cartan form, we have the Maurer-Cartan equation, $d\Omega = -\Omega \wedge \Omega$ which remains valid restricted to \mathcal{F}^1 . Thus we calculate that on \mathcal{F}^1 , $0 = d\omega^a = -\omega_j^a \wedge \omega^j$ which, by the Cartan Lemma, implies that $\omega_j^a = h_{jk}^a \omega^k$ for some functions $h_{jk}^a = h_{kj}^a$ well defined on \mathcal{F}^1 . Although these functions vary in the fiber, the tensor

$$II_x := h_{jk}^a \omega^j \omega^k \otimes e_a$$

descends to be a well defined element of $S^2 T_x^* M \otimes N_x M$, called the *second fundamental form* of M at x . M is said to be *minimal* if $\text{trace}_g(II) = 0$, where g is the induced Riemannian metric on M . Such M are locally of least volume in the sense explained before for surfaces.

An exterior differential system for submanifolds of Euclidean space. An eds for lifts of submanifolds in \mathbb{E}^{n+s} may be defined on $\mathcal{F} \times \mathbb{R}^{\binom{n+1}{2}s}$ where $\mathbb{R}^{\binom{n+1}{2}s}$ has coordinates $h_{ij}^a = h_{ji}^a$, $1 \leq i, j \leq n$, $3 \leq a, b \leq 2 + s$. The system is

$$\begin{aligned} (\text{sub.1}) \quad \mathcal{I} &= \{\omega^a, \theta_i^a := \omega_i^a - h_{ij}^a \omega^j\}_{\text{diff}} \\ \Omega &= \omega^1 \wedge \dots \wedge \omega^n \end{aligned}$$

This is the analog of the canonical contact system on J^1 . To study submanifolds satisfying a differential equation, we restrict to the submanifold of $\mathcal{F} \times \mathbb{R}^{\binom{n+1}{2}s}$ where the equation is satisfied.

An exterior differential system for minimal surfaces. An eds for lifts of minimal surfaces in \mathbb{E}^{2+s} may be defined on $\Sigma \subset \mathcal{F} \times \mathbb{R}^{3s}$ where \mathbb{R}^{3s} has coordinates $h_{ij}^a = h_{ji}^a$, $1 \leq i, j \leq 2$, $3 \leq a, b \leq 2 + s$ and where Σ is defined by the s equations $h_{11}^a + h_{22}^a = 0$.

Remark. Note that the condition $h_{11} + h_{22}$ is significantly simpler than the equation for minimal surfaces presented in example 1. This illustrates the advantage of frames.

Our system is the restriction of (sub.1) to Σ . Differentiating, we have

(sub.2)

$$\begin{aligned} d\omega^a &\equiv 0 \bmod \{\omega^a, \theta_i^a\} \\ d\begin{pmatrix} \theta_1^a \\ \theta_2^a \end{pmatrix} &\equiv \begin{pmatrix} 2h_{21}^a \omega_1^2 - dh_{11}^a & -(2h_{11}^a \omega_1^2 - dh_{12}^a) \\ -(2h_{11}^a \omega_1^2 - dh_{12}^a) & -(2h_{21}^a \omega_1^2 - dh_{11}^a) \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \bmod \{\omega^a, \theta_i^a\} \\ &\equiv \begin{pmatrix} \pi_1^a & \pi_2^a \\ \pi_2^a & -\pi_1^a \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \bmod \{\omega^a, \theta_i^a\} \end{aligned}$$

where the last line defines the one-forms π_1^a, π_2^a .

Here there are Cauchy characteristic vector fields. The distribution they span is $\{\omega^i, \theta^a, \pi_i^a\}^\perp$. (“ $^\perp$ ” refers to the annihilator in the dual space.)

If we reverse the order of the equations in (sub.2) and write

$$d\begin{pmatrix} \theta_2^a \\ \theta_1^a \end{pmatrix} \equiv \begin{pmatrix} \pi_2^a & -\pi_1^a \\ \pi_1^a & \pi_2^a \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \bmod \{\omega^a, \theta_i^a\}$$

the system ‘looks like’ the Cauchy-Riemann equations as presented in example 2.

On $T^*\mathbb{C}\Sigma$ let

$$\omega := \omega^1 + i\omega^2, \quad \theta^a := \theta_2^a + i\theta_1^a, \quad \pi^a := \pi_1^a + i\pi_2^a$$

Noting that

$$\begin{aligned} d\omega &= i\omega_1^2 \wedge \omega \\ d\theta &= -\pi \wedge \omega \\ d\pi &= 2\pi \wedge \omega_1^2 + 2(h_{11} + ih_{21})K\omega \wedge \bar{\omega} \end{aligned}$$

we see the almost complex structure is integrable, and moreover the second equation implies $d\theta \wedge \theta \neq 0$. Thus we have a complex contact structure. From this we recover the result that minimal surfaces in \mathbb{E}^{2+s} are given by holomorphic data. To recover an actual coordinate presentation, one would have to work more. In summary

Theorem (see e.g. [Ch, p. 19]). *Any minimal surface $M^2 \subset \mathbb{E}^{2+s}$ is the transform of a holomorphic contact curve.*

There is no nice description of all local solutions to the minimal submanifold system for $n > 2$. To understand the situation better, it is often useful to look at special classes of solutions, e.g. solutions with symmetry. (see e.g. [HsL]). Taking a broad view of what constitutes “symmetry”, it has something to do with reducing the complexity of the problem via the action of some group. For example, if n and s are even, a natural sub-class of minimal submanifolds of \mathbb{E}^{n+s} are the complex submanifolds (we will see in a minute that these are all minimal). I will describe some special classes of minimal submanifolds shortly, but first I’ll take a short detour to discuss minimizing submanifolds.

Calibrated submanifolds (Harvey-Lawson [HaL]).

Let (X^{n+s}, g) be a Riemannian manifold. A submanifold $M^n \subset X$ is said to be *minimizing* if it is minimal and such that for all $x \in M$, $\tilde{U}_x = M$, where \tilde{U}_x is the open neighborhood described in example 1. While being minimal is a local property, being minimizing is a global matter and in general is difficult to prove. Harvey and Lawson, building on Wirtinger’s and Federer’s observations about the Kähler form, made the following definition:

Let (X^{n+s}, g) be a Riemannian manifold. A *calibration* $\phi \in \Omega^n(X)$ is an n -form having the properties:

i. $d\phi = 0$

ii. $\phi_x(E) \leq 1$ for all $E \in G(n, T_x X)$, for all $x \in X$ (where here $G(n, T_x X)$ denotes the Grassmannian of unit n -planes).

Theorem [HaL]. Let (X^{n+s}, g) be a Riemannian manifold with a calibration $\phi \in \Omega^n(X)$. Let $i : M^n \rightarrow X^{n+s}$ be such that $i^*(\phi) = d\text{vol}_M$. Then M is minimizing.

Proof. Let $U \subset M$ be any open neighborhood and let $V^n \subset X$ be such that $\partial U = \partial V$. Then we have

$$\text{volume}(U) = \int_U \phi = \int_V \phi \leq \text{volume}(V)$$

where the second equality is due to Stokes' theorem. \square

Given a calibration ϕ , let $\text{face}_x(\phi) \subset G(n, T_x X) = \{E \in G(n, T_x X) | \phi(E) = 1\}$. If $X = \mathbb{E}^{n+s}$ and ϕ has constant coefficients, one can talk about the face of ϕ in $G(n, n+s)$. (Here, and in what follows, $G(n, n+s)$ denotes the orthogonal Grassmannian.) Note that if $\gamma(M) \subset \text{face}(\phi)$ then M is minimizing.

In summary:

$$\begin{aligned} M \text{ is minimal} &\leftrightarrow \text{conditions on the} \\ &\quad \text{derivative of } \gamma \\ M \text{ is minimizing} &\leftarrow \text{conditions on the} \\ &\quad \text{image of } \gamma \end{aligned} \quad \begin{aligned} &\leftrightarrow \text{second order system of pde} \\ &\leftrightarrow \text{first order system of pde} \end{aligned}$$

Examples: In [HaL], Harvey and Lawson study the following faces of calibrations:

0. The $SU(m+r)$ orbit of a complex m -plane, which is the complex Grassmannian $G(\mathbf{C}^m, \mathbf{C}^{m+r}) \subset G_{2m, 2(m+r)}$. (Already studied by Wirtinger and Federer.)
1. The $SU(n)$ orbit of a real n -plane in \mathbb{E}^{2n} , called the *special Lagrangian face* $\subset G(n, 2n)$.
2. The G_2 orbit of an associative 3-plane in \mathbb{E}^7 , called the *associative face* $\subset G(3, 7)$.
3. The G_2 orbit of a coassociative 4-plane in \mathbb{E}^7 , called the *coassociative face* $\subset G(4, 7)$.
4. The $Spin(7)$ orbit of a quaternionic 4-plane in \mathbb{E}^8 , called the *Cayley face* $\subset G(4, 8)$.

Harvey and Lawson proved all these faces are *involutive* in the sense of the Cartan-Kähler theorem, that is, that they admit a large class of minimizing submanifolds.

There are not many examples or classes of examples of minimal submanifolds studied explicitly. To produce some new classes of examples, I made the following definition, generalizing the notion of the faces of calibrations:

Definition. An *m -subset* is a submanifold $\Sigma \subset G(n, n+s)$ with the property that any $M^n \subset \mathbb{E}^{n+s}$ with $\gamma(M) \subset \Sigma$ is necessarily a minimal submanifold. Given an m -subset Σ , we call any M with $\gamma(M) \subset \Sigma$ a *Σ -manifold*.

An m -subset Σ corresponds to a first order system of pde that implies the minimal submanifold system. What follows is a description of an m -subset that is not a face:

If we consider \mathbb{E}^5 as the space of 3×3 traceless-symmetric matrices, then $SO(3)$ acts on \mathbb{E}^5 by conjugation (i.e. given $x \in \mathbb{E}^5$ and $g \in SO(3)$, the action is $x \mapsto gxg^{-1}$). Let $\rho : SO(3) \rightarrow SO(5)$ denote the representation. We will use this $\rho(SO(3))$ action as our symmetry. Since $\rho(SO(3))$ acts on \mathbb{E}^5 , it acts on the linear subspaces, in particular on the Grassmannian $G(3, 5)$ (here and in what follows I mean the orthogonal Grassmannian). There are two special orbits of dimension two (where a maximal torus preserves the 3-plane). Let $\Sigma \subset G(3, 5)$ be one such orbit. (Notational apology- this Σ has nothing to do with the spaces we define the exterior differential systems on.) When one sets up the corresponding eds one finds the following:

Theorem [L1]. Σ is an m -subset, that is any $M^3 \subset \mathbb{E}^5$ with $\gamma(M) \subset \Sigma$ is minimal.

Let \mathcal{L} denote the space of all $\rho(SO(3))$ - weight zero lines in \mathbb{E}^5 (that is, the weight zero lines through the origin and their translates). Then \mathcal{L} is a complex contact manifold, and Σ -manifolds are exactly the transforms of the complex contact curves in \mathcal{L} .

One may also think of the weight zero lines through the origin as the lines through rank one matrices.

How did I find the space \mathcal{L} ? \mathcal{L} corresponds to \mathcal{F} quotiented by the Cauchy characteristics. The picture is as follows:

$$\begin{array}{ccc} & \mathcal{F} & \\ \mathcal{L} & & \mathbb{E}^5 \end{array}$$

Note that the transform of a point in \mathcal{L} is a line in \mathbb{E}^5 . One can choose coordinates on \mathcal{L} and compatible coordinates on \mathbb{E}^5 to get an explicit Weierstrass-type presentation formula:

Theorem [L1, 5.1]. *Given a holomorphic function of one variable $h(z)$, setting $w = h(z)$, $y = \frac{dh}{dz}$ one obtains a minimal 3-manifold in \mathbb{E}^5 given in coordinates by*

$$\begin{aligned} x^0 &= \frac{1 - 4|z|^2 + |z|^4}{2\sqrt{3}(1 + |z|^2)^2} t + \frac{2(-2 + 2|z|^2 + |z|^4)}{\sqrt{3}(1 + |z|^2)^4} \operatorname{Re}(\bar{z}^2 w) - \frac{-5 + 2|z|^2 + |z|^4}{2\sqrt{3}(1 + |z|^2)^3} \operatorname{Re}(\bar{z}y) \\ x^1 + ix^2 &= \frac{z(1 - |z|^2)}{(1 + |z|^2)^2} t - \frac{2\bar{z}|z|^2(2 + |z|^2)}{(1 + |z|^2)^4} w - \frac{2z^3}{(1 + |z|^2)^4} \bar{w} - \frac{1 - 2|z|^2 - |z|^4}{2(1 + |z|^2)^3} y + \frac{z^2}{(1 + |z|^2)^3} \bar{y} \\ x^3 + ix^4 &= \frac{z^2}{(1 + |z|^2)^2} t + \frac{1 + 4|z|^2 + 2|z|^4}{(1 + |z|^2)^4} w - \frac{z^4}{(1 + |z|^2)^4} \bar{w} - \frac{z(|z|^2 + 2)}{2(1 + |z|^2)^3} y + \frac{z^3}{2(1 + |z|^2)^3} \bar{y} \end{aligned}$$

Remark: These are not all the solutions, only solutions having invertible Gauss maps are obtained. For example 3-planes are not among the above solutions. One could derive a more general formula, but it would involve integrals, as in the classical formula. (Conversely, one can write a Weierstrass type formula in the classical case without an integral for a large class of minimal surfaces in \mathbb{E}^3 .)

One could ask about the simplest such solution, the transform of the function $h(z) \equiv 0$.

Proposition [L2, 5.2]. *The minimal submanifold given by $h(z) \equiv 0$ is the cone over the real Veronese, that is, the set of rank one matrices whose repeated eigenvalue is positive.*

Some more examples of m -subsets:

Theorem [L1,2.1]. *Let $\rho : SO(3) \rightarrow SO(2n+1)$ be an irreducible representation and let $\Sigma \subset G(k, 2n+1)$ be any two dimensional $\rho(SO(3))$ orbit. Then Σ is an m -subset.*

Recalling that $\mathfrak{so}(3) = \mathfrak{su}(2)$ another generalization is to $SU(m)$ orbits:

Theorem [L2, 3.10]. *Let V^N denote any irreducible real $SU(n+1)$ module other than the adjoint. Then the orbits Σ of certain codimension 2-planes are m -subsets.*

The Σ -manifolds may be constructed in a manner analogous to the situation above. One constructs a space of special linear spaces, \mathcal{L} , which is a complex contact manifold. The Σ -manifolds are the transforms complex n -folds in \mathcal{L} that solve the differential system induced from \mathcal{F} .

The Σ -manifolds corresponding to the simplest solutions of the differential system on \mathcal{L} are homogeneous cones, in fact the $SU(n+1)$ orbits of special $(N - 2n - 2)$ -planes in V .

One can also construct inhomogeneous examples:

Recall that $G(n, n+2)$ and $G(2, 2+s)$ are complex manifolds, in fact quadric hypersurfaces $Q^n \subset \mathbb{CP}^{n+1}$, $Q^s \subset \mathbb{CP}^{s+1}$.

Bryant (personal communication) proved that complex submanifolds $\Sigma \subset G(2, 2+s)$ whose tangent spaces contain no decomposable vectors are m -subsets. (While it was known classically that a surface was minimal iff its Gauss map was holomorphic, Bryant proved it was sufficient that the Gauss image was a complex submanifold satisfying a transversality condition.)

Proposition [L2, 3.8]. *Complex submanifolds $\Sigma \subset G(n, n+2)$ whose tangent spaces contain no decomposable vectors are m -subsets.*

Corollary [L2, 3.9]. *The generic complex submanifold of $G(n, n+2)$ of complex dimension $[\frac{n}{2}]$ is an m -subset.*

One might hope to get new m -subsets by deforming faces. One can rephrase [L2, 3.8] as:

Theorem [L2]. *The complex Grassmannians $G(\mathbb{C}^{m-1}, \mathbb{C}^m) \subset G(2(m-1), 2m)$ admit deformations in the category of m -subsets.*

A deeper result is the following:

Theorem [L2, 3.1, 3.1*]. *All the faces 0-4 studied by Harvey and Lawson except $G(\mathbb{C}^1, \mathbb{C}^m)$ and $G(\mathbb{C}^{m-1}, \mathbb{C}^m)$ are rigid in the larger category of m -subsets.*

Note that the deformation theory was not previously known even in the restricted category of faces.

Because it will be useful for our next lecture, let me now explain how one looks for m -subsets.

Recall that $T_E G(n, n+s) \simeq E^* \otimes E^\perp$. Given a submanifold $i : M^n \rightarrow \mathbb{E}^{n+s}$, we have its *Gauss map* $\gamma : M \rightarrow G(n, n+s)$, where $x \mapsto T_x M$. Consider the derivative of the Gauss map, $d\gamma_x \in T_x^* M \otimes T_{T_x M} G(n, n+s) = T_x^* M \otimes (T_x^* M \otimes N_x M)$. Now $d\gamma_x \simeq II_x$, and since the second fundamental form is symmetric, we have $d\gamma_x \in (S^2 T_x^* M \otimes N_x M)$.

Now if $\Sigma \subset G(n, n+s)$ is any submanifold, then $T_E \Sigma \subset E^* \otimes E^\perp$ is some linear subspace. If $\phi : M \rightarrow \Sigma$ is any map, then $d\phi_x \in T_x^* M \otimes T_{\phi(x)} \Sigma$, so if $\gamma(M) \subset \Sigma$, then we must have for all $x \in M$ that

$$(\gamma) \quad d\gamma_x \in (S^2 T_x^* M \otimes N_x M) \cap (T_x^* M \otimes T_{T_x M} \Sigma).$$

where the first condition is because $d\gamma_x$ is a second fundamental form and the second because the image lies in Σ .

More generally, given a linear subspace $A \subset V^* \otimes W$, define the *prolongation* of A , $A^{(1)}$, by

$$A^{(1)} := (A \otimes V^*) \cap (W \otimes S^2 V^*)$$

We may think of A as specifying a first order constant coefficient system of pde for mappings $f : V \rightarrow W$, namely the system whose solutions are those mappings f such that $Jac(f)_x \in A$ for all $x \in V$. In other words, A consists of the admissible first order terms for the Taylor series of f . With this perspective, $A^{(1)}$ has the interpretation of the admissible second order terms in the Taylor series. We will see that $A^{(1)}$ also may be identified with the space of integral elements of the corresponding eds at each point. The condition (γ) is that $d\gamma_x \in (T_{T_x M} \Sigma)^{(1)}$.

The fundamental observation for the study of m -subsets [L2]. *If $\Sigma \subset G(n, n+s)$ is such that for all $E \in \Sigma$,*

$$(T_E \Sigma)^{(1)} \subseteq S^2 E_0^* \otimes E^\perp$$

where $S^2 E_0^*$ denotes the traceless elements, then Σ is an m -subset.

Thus the infinitesimal version of the problem to find examples of m -subsets is to find first order constant coefficient systems that imply the Laplace system. (In practice, one restricts to *involutive* systems, as will be defined in the next lecture.)

The theorems above were proved first by studying the linear problem, that is studying first order constant coefficient systems of pde that imply Laplace's system and then by setting up the appropriate exterior differential system for m -subsets whose tangent spaces (linearized system of pde) correspond to solutions of the linearized system.

Problems for lecture 2.

1. Define a canonical system for all n -dimensional submanifolds of \mathbb{E}^{n+s} on \mathcal{F} . Show that the system (sub.1) is the prolongation of this system. What are the Cauchy Characteristic vector fields of this system? Give a geometric interpretation of the quotient space $\mathcal{F}/\text{leaves}$.
2. Write down an eds for surfaces in \mathbb{E}^3 with Gauss curvature identically one. What is the dimension of the space of integral elements at a point?
3. Write down an eds for surfaces of revolution in \mathbb{E}^3 with Gauss curvature identically one. Find all integral manifolds of the system.
4. Show that the eds for constant mean curvature one surfaces in hyperbolic three space is equivalent to the Cauchy-Riemann system. What about constant mean curvature in \mathbb{E}^3, S^3 ?

LECTURE 3: THE CARTAN ALGORITHM VIA THE ISOMETRIC EMBEDDING PROBLEM

Example 4, The isometric embedding problem. Let (M^n, g) be a patch of an analytic Riemannian manifold. What (if any) are the isometric embeddings $M^n \rightarrow \mathbb{E}^{n+r}$?

The question is stated a little vaguely- we first want to know if, given r , there exist any embeddings, and then, if so, “how many”?

To begin, let \mathcal{F}_M and $\mathcal{F}_{\mathbb{E}^{n+r}}$ be the respective frame bundles with $1 \leq i, j \leq n$, $n+1 \leq \mu, \nu \leq n+r$. We use η to denote forms on \mathcal{F}_M and ω forms on $\mathcal{F}_{\mathbb{E}^{n+r}}$. On \mathcal{F}_M we have equations:

$$\begin{aligned} dx &= \eta^i e_i \\ de_i &= \eta_i^j e_j \\ d\eta^j &= -\eta_k^j \wedge \eta^k \\ d\eta_j^i &= -\eta_l^i \wedge \eta_j^l + R_{jkl}^i \eta^k \wedge \eta^l \end{aligned}$$

where R_{jkl}^i are well defined functions on \mathcal{F}_M , the coefficients of the Riemann curvature tensor.

If $\Gamma \subset \mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}}$ is the graph of an isometric embedding, it is natural to demand that on Γ $\text{span}\{\omega^i\} = \text{span}\{\eta^i\}$ and that $\omega^\mu = 0$. (Here and in what follows, we commit a standard abuse of notation, what we really mean that if the inclusion map is $i : \Gamma \rightarrow \mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}}$, that $i^*(\omega^\mu) = 0$ etc...). In addition, it is natural to eliminate some of the Cauchy characteristics by requiring the bases to line up, that is requiring $\omega^i = \eta^i$ on Γ . With these requirements, we obtain, on $\mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}}$, the Pfaffian system with independence condition

$$\begin{aligned} \mathcal{I}_0 &= \{\omega^\mu, \omega^j - \eta^j\} \\ \Omega &= \eta^1 \wedge \dots \wedge \eta^n \end{aligned}$$

From experience, it is also clear what to do next, namely differentiate (again, we have defined a distribution and now we want to see how the different planes of the distribution ‘fit together’). We find

$$(iso.1) \quad d\omega^\mu \equiv -\omega_j^\mu \wedge \eta^j \pmod{\mathcal{I}_0}$$

$$(iso.2) \quad d(\omega^i - \eta^i) \equiv -(\omega_j^i - \eta_j^i) \wedge \eta^j \pmod{\mathcal{I}_0}$$

Using the independence condition and the Cartan Lemma, we see that for (iso.1), (iso.2) to hold, there must be functions u_{jk}^i, h_{jk}^μ , defined on Γ , symmetric in their lower indices, such that, on Γ ,

$$(\omega_j^i - \eta_j^i) \equiv u_{jk}^i \eta^k \pmod{\mathcal{I}_0}$$

$$\omega_j^\mu \equiv h_{jk}^\mu \eta^k \pmod{\mathcal{I}_0}$$

In fact $u_{jk}^i = 0$ for all i, j, k (exercise - why?). So we have $\dim A^{(1)} = r \binom{n+1}{2}$. At this point we need to perform Cartan's test to see if our initial value problem is well posed. Informally, write out tableau as

$$A = \begin{pmatrix} \omega_j^\mu \\ \omega_j^i - \eta_j^i \end{pmatrix}$$

an $(r+n) \times n$ matrix of one-forms. Note that $\dim A = nr + \binom{n}{2}$. To perform Cartan's test (as explained below), let $L_j \subset V^*$ be a generic linear subspace of dimension j and let $A_k = A \cap (L_{n-k} \otimes W)$. We need to calculate $\dim A_k$. If our bases are sufficiently generic, $\dim A_{n-k}$ is just the number of linearly independent forms in the first k columns. We get $\dim A_{n-k} = rk + (n-1) + (n-2) + \dots + (n-k)$. Cartan's test says that the initial value problem is well posed, or that the system is *involutive* if $\dim A^{(1)} = \dim A + \sum_{k=1}^{n-1} \dim A_k$. Here

$$\sum_k \dim A_k = r \binom{n+1}{2} + n(n-1) + (n-1)(n-2) + \dots + 2(1) > r \binom{n+1}{2} = \dim A^{(1)}$$

so the system is not involutive. Thus we must prolong, which amounts to adding the space of integral elements to our ambient manifold and the forms $\omega_j^\mu - h_{jk}^\mu \eta^k, \omega_j^i - \eta_j^i$ to our ideal. That is let

$$\tilde{\Sigma} = \mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}} \times \mathbb{R}^{r+\binom{n+1}{2}}$$

and let

$$\mathcal{I} = \{\omega^\mu, \omega^j - \eta^j, \omega_j^i - \eta_j^i, \omega_j^\mu - h_{jk}^\mu \eta^k\}$$

The h_{jk}^μ correspond to the coefficients of the second fundamental form of the submanifold, so what we have done here is to add second derivatives as independent variables. We have now rigged things such that on integral elements

$$\begin{aligned} d\omega^\mu &\equiv 0 \pmod{\mathcal{I}} \\ d(\omega^i - \eta^i) &\equiv 0 \pmod{\mathcal{I}} \end{aligned}$$

This is always true when one prolongs, the old forms automatically will have zero derivative modulo the new ideal. Now we must compute the other derivatives.

$$d(\omega_j^i - \eta_j^i) \equiv (R_{jkl}^i - \Sigma_\mu (h_{ik}^\mu h_{jl}^\mu - h_{il}^\mu h_{jk}^\mu)) \eta^k \wedge \eta^l \pmod{\mathcal{I}}$$

Here we are presented with a similar problem as in example 1, the right hand side involves independent forms in the independence condition. There is no way to have $d(\omega_j^i - \eta_j^i) = 0$ on an integral manifold unless all the functions

$$(iso.3) \quad R_{jkl}^i - \Sigma_\mu (h_{ik}^\mu h_{jl}^\mu - h_{il}^\mu h_{jk}^\mu)$$

are identically zero. In order to ensure this, we start over yet again, this time on the submanifold $\Sigma' \subset \Sigma$ defined by requiring the functions in (iso.3) to be zero. (iso.3) is an example of the appearance of terms representing *torsion*. There is a danger that when one restricts to the submanifold of the original manifold defined by setting the torsion equal to zero that the independence condition may no longer hold. When this happens there are no integral manifolds to the system (as in example 1, case 2b). Fortunately here this is not a concern, because we may view the additional equations as cutting down the $\mathbb{R}^{r+\binom{n+1}{2}}$ factor and not touching \mathcal{F}_M in the product, which is where the independence condition lies.

Now we continue to take derivatives:

$$d(\omega_j^\mu - h_{jk}^\mu \omega^k) \equiv (dh_{jk}^\mu - h_{jk}^\nu \omega_\nu^\mu + h_{jl}^\mu \omega_k^l + h_{kl}^\mu \omega_j^l) \wedge \omega^k \pmod{\mathcal{I}}$$

To simplify notation, let $\pi_{jk}^\mu = dh_{jk}^\mu - h_{jk}^\nu \omega_\nu^\mu + h_{jl}^\mu \omega_k^l + h_{kl}^\mu \omega_j^l$. Note that Σ' has a coframing by $\eta^i, \eta_j^i, \omega^i, \omega^\mu, \omega_j^\mu, \pi_{jk}^\mu, \omega_j^i, \omega_\nu^\mu$. The last two groups of forms are dual to infinitesimal rotations in the tangent and normal bundles, and as before, these infinitesimal rotations are Cauchy characteristic vector fields. That is, the Cauchy characteristic distribution is

$$\{\eta^i, \eta_j^i - \omega_j^i, \omega^i, \omega^\mu, \omega_j^\mu, \pi_{jk}^\mu\}^\perp.$$

Now we are in a situation where we have relations among differential forms and their derivatives at a point.

In the determined case, if one applies the Cartan test again at a general point of Σ' , one finds that the system is in *involution* (as defined below) and one recovers the BCJS theorem stated earlier.

In the overdetermined cases, for a general metric there will be torsion and if one restricts to the submanifold $\Sigma'' \subset \Sigma'$ where the torsion is zero, one finds that the independence condition no longer holds so that there are no integral manifolds. However, for certain special metrics, there are solutions in some overdetermined cases, e.g. the sphere $S^n \subset \mathbb{E}^{n+1}$.

Here are some additional results on isometric embeddings in the overdetermined case.

Theorem (Cartan) [C1,C2]. *Let (M^n, g) be a (patch of a) flat Riemannian manifold, then M admits no linearly full local isometric embedding to \mathbb{E}^m for $n < m < 2n$.*

M does admit linearly full local isometric embeddings to \mathbb{E}^{2n} , and these depend locally on $\binom{n}{2}$ functions of two variables.

Theorem (Cartan) [C1,C2]. *Let (M^n, g) be a (patch of a) Riemannian manifold with constant sectional curvature -1 , then M admits no local isometric embedding to \mathbb{E}^m for $m < 2n - 1$.*

M does admit local isometric embeddings to \mathbb{E}^{2n-1} , and these depend locally on $n(n-1)$ functions of one variable, which is the most possible for any Riemannian n -fold admitting local isometric embeddings to \mathbb{E}^{2n-1} .

In [BBG], Cartan's theorem was generalized to a larger class of Riemannian metrics, called *quasi-hyperbolic metrics*. In [IvL], Ivey and I generalized their results further to a class of metrics we call *quasi- κ -curved metrics*:

Definition: Let (M^n, \tilde{g}) be a Riemannian manifold. We will say \tilde{g} is a *quasi- κ -curved metric* if there exists a smooth positive definite quadratic form Q on M such that for all $x \in M$

$$R_x = -\gamma(Q_x, Q_x) + (\kappa + 1)\gamma(\tilde{g}_x, \tilde{g}_x)$$

where $\gamma : S^2 T^* \rightarrow S^2(\Lambda^2 T^*)$ denotes the algebraic Gauss mapping and R_x the Riemann curvature tensor.

Following [BBG] (and the real work was done by them and Cartan), we showed:

Theorem [IvL, A]. *Let (M^n, g) , $n \geq 3$, be a quasi- κ -curved Riemannian manifold with form Q . Let $X^{2n-1}(\kappa + 1)$ be a space form with constant sectional curvature $\kappa + 1$. Then there exist local isometric embeddings $M^n \hookrightarrow X^{2n-1}(\kappa + 1)$, with local solutions depending on $n^2 - n$ functions of one variable, if and only if ∇Q is a symmetric cubic form on M and $\nabla Q = L \cdot Q$ for some linear form $L \in \Omega^1(M)$.*

In [BBG], quasi-hyperbolic n -folds were characterized as space-like hypersurfaces in \mathbb{L}^{n+1} . We observed that quasi- κ -curved n -folds arise naturally as spacelike hypersurfaces of a Lorentzian sphere having radius $\frac{a}{\sqrt{\kappa+1}}$ in \mathbb{L}^{n+2} and we determined which of these admits optimal isometric embeddings [IvL, theorem B].

Our original motivation was to look for interesting classes of minimal isometric embeddings, what we actually ended up proving were several non-existence and rigidity results [IvL, theorems C,D,E].

CARTAN'S ALGORITHM FOR LINEAR PFAFFIAN SYSTEMS

Given (I, J) , a linear Pfaffian system on Σ .

Step 1. Take a coframing of Σ adapted to the filtration $I \subset J \subset T^*\Sigma$ with forms $(\theta^a, \omega^i, \pi^\epsilon)$, which we assume to be generic among such adapted coframings. Fix $x \in \Sigma$ and write $V^* = (J/I)_x$, $W^* = I_x$, $v^i = \omega_x^i$, $w^a = \theta_x^a$ and v_i, w_a the corresponding dual basis vectors.

Step 2. Calculate $d\theta^a$. Since the system is linear Pfaffian, $d\theta^a$ is of the following form:

$$d\theta^a \equiv A_{\epsilon i}^a \pi^\epsilon \wedge \omega^i + T_{ij}^a \omega^i \wedge \omega^j \bmod I$$

The forms π_i^a used previously were $\pi_i^a = A_{\epsilon i}^a \pi^\epsilon$. If one wishes to work on a quotient space, the Cauchy characteristic distribution is $\{\theta^a, \omega^i, \pi_i^a\}^\perp$.

Define the *tableau*

$$A = A_x := \{A_{\epsilon i}^a v^i \otimes w_a \subseteq V^* \otimes W \mid 1 \leq \epsilon \leq r\}$$

Let δ denote the natural skew symmetrization map $\delta : W \otimes V^* \otimes V^* \rightarrow W \otimes \Lambda^2 V^*$ and let

$$H^{0,2}(A_x) := W \otimes \Lambda^2 V^* / \delta(A_x \otimes V^*)$$

The *torsion* of (I, J) at x is

$$[T_x] := [T_{ij}^a w_a \otimes v^i \wedge v^j] \in H^{0,2}(A_x).$$

Note that in our examples the torsion was (2.2) in example 1, and (iso.3) in example 4.

Step 3. If $[T] \neq 0$, then start again on $\Sigma' \subset \Sigma$ defined by the equations $[T] = 0$. In practice, since one works infinitesimally, one only uses the equations $[dT] = 0$ and checks what relations this forces on the forms one was using.

Step 4. Assuming $[T] = 0$, Let $A_{n-j} := A \cap (\text{span}\{v^1, \dots, v^j\} \otimes W)$. Let $A^{(1)} := (A \otimes V^*) \cap (W \otimes S^2 V^*)$, the *prolongation* of the tableau A .

Proposition (Cartan) [BCG³, p120]. $\dim A^{(1)} \leq \dim A + \dim A_1 + \dots + \dim A_{n-1}$.

We say A is *involutive* if equality holds in the proposition.

In practice it is more convenient to formulate Cartan's inequality as follows: Let the *characters* of A , s_j , be the integers defined inductively by $s_1 + \dots + s_j = \dim A_{n-j}$. (In practice this is just the number of independent entries appearing in the first j columns of A , when A is written as a subspace of the $\dim V \times \dim W$ matrices.) With this formulation, Cartan's proposition becomes

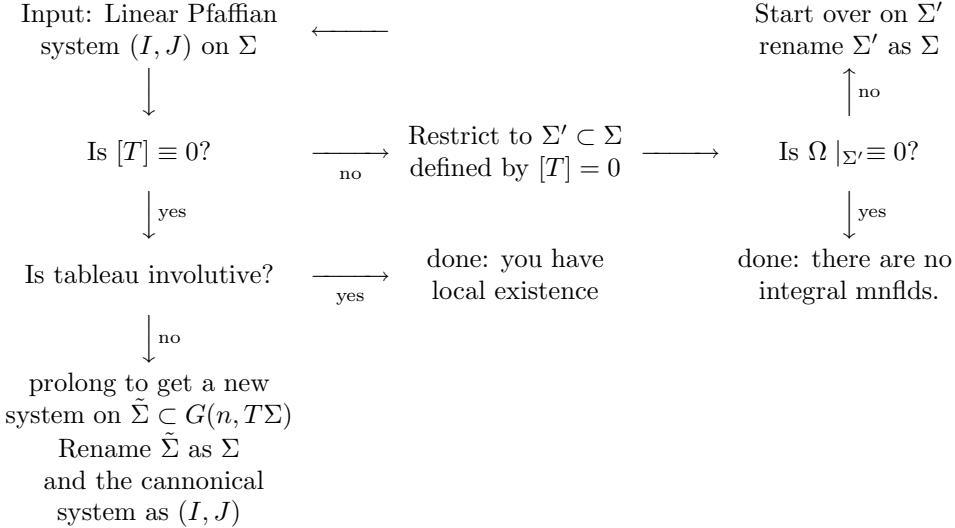
$$\dim A^{(1)} \leq s_1 + 2s_2 + \dots + ns_n.$$

The Cartan-Kähler theorem for linear Pfaffian systems. *Let (I, J) be a linear Pfaffian system on Σ and let $x \in \Sigma$ be a point where all numerical invariants defined above are locally constant. If $[T_x] = 0$ and A_x is involutive, then there are local solutions to an initial value problem depending on s_p functions of p variables, where s_p is the last nonzero character.*

The Cartan-Kähler theorem is a generalization of the Cauchy-Kowalevski theorem, and its proof involves reducing the problem of finding integral manifolds to a series of Cauchy problems.

Step 5. If A is not involutive, one needs to start over on a larger space. Invariantly, one prolongs as described in lecture 1. In practice this amounts to enlarging Σ to include the elements of $A^{(1)}$ as independent variables, and adding differential forms to the ideal $\theta_i^a := A_{\epsilon i}^a \pi^\epsilon - p_{ij}^a \omega^j$, where $p_{ij}^a v^i \otimes w_a \in A^{(1)}$. In our isometric embedding example, we added h_{ij}^μ as new variables and the forms $\omega_j^i - \eta_j^i, \omega_j^\mu - h_{jk}^\mu \omega^k$ to the ideal.

In summary, we have the following flowchart:



Problems for lecture 3 and beyond.

1. Determine the space of integral manifolds for problems 1 and 2 from the problems for lecture 2.
2. Perform Cartan's test for the Cauchy Riemann equations. Interpret the result in terms of the holomorphic function $f(z)$.
3. Characterize the surfaces $M^2 \subset \mathbb{E}^3$ such that all points of M are umbilic points (that is, the two eigenvalues of the second fundamental form are everywhere equal).
4. Prove that given any nonplanar minimal surface $M^2 \subset \mathbb{E}^3$, there is a one-parameter family of noncongruent deformations that preserve the induced Riemannian metric and minimality.
5. Calculate Cartan's test for the case $\Sigma = J^1(\mathbb{R}^n, \mathbb{R}^s)$ with its canonical system and for its prolongation. (Of course the answer should be that solutions depend on s functions of n variables.)
6. The Grassmannian $G(n, n+s)$ of oriented orthogonal n -planes in \mathbb{R}^{n+s} has a natural Riemannian metric on it (such that the forms ω_i^a form an orthonormal framing). Thus given $i : M^n \rightarrow \mathbb{E}^{n+s}$ we can compare the metric induced by i with that induced by the Gauss map γ . Suppose that the two metrics agree pointwise up to a constant, i.e. $g_i = \lambda g_\gamma$, where λ is some function.
 - a. Show that in fact λ must be constant.
 - b. Show that such M are Einstein, that is $Ricci_g = cg$ for some constant c .
 - c. Classify all such hypersurfaces, i.e. the case $s = 1$.
 - d. Classify the case $n = 3, s = 2$. In particular show there are "many" such. How many?
7. Show that the eds for special Lagrangian submanifolds is involutive. Integral manifolds depend on how many functions of how many variables?
8. (For exceptional groups enthusiasts) Same question as 6 for the other faces of calibrations defined by Harvey and Lawson.

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